UNIT-3. Group Actions.

Definition → Let G be a group and S be a non-empty set. A group action of G on S is a map $\cdot : G \times S \to S$ defined by for each $x \in S$

(i) $e \cdot x = x$, $e \in G$, being the identity of G.

(ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$, for all $g_1, g_2 \in G$.

We say that G acts on S and S is a G-set.

Examples:

① Consider G a subgroup of the group of permutations on some set S; $G \leq Perm(S)$.
Define a group action of G on S by
$$\sigma \cdot x = \sigma(x), \ \forall x \in S, \forall \sigma \in G. \quad \boxed{\sigma : S \to S \text{ is a bijection.}}$$
Let i_G be the identity map $\in G$.
Here i_G is the identity permutation on S.
Now $i_G \cdot x = i_G(x) = x, \ \forall x \in S$.
Also, if $\sigma_1, \sigma_2 \in G$, then
$$\begin{aligned}(\sigma_1 \sigma_2) \cdot x &= (\sigma_1 \sigma_2)(x), \ \forall x \in S \\ &= \sigma_1(\sigma_2(x)) \\ &= \sigma_1 \cdot (\sigma_2(x)) = \sigma_1 \cdot (\sigma_2 \cdot x).\end{aligned}$$
$\therefore$ S is a G-Set.

② Suppose $H \leq G$, then H acts on G according to (i) $h \cdot x = hx$, $\forall h \in H$ and $\forall x \in G$.
Naturally, $e \cdot x = ex = x; \forall x \in G$, & $e \in H$.
and
$$\begin{aligned}(ab) \cdot x &= (ab)x = a(bx), \text{ by associative property} \\ &= a \cdot (bx) \\ &= a \cdot (b \cdot x), \ \forall a, b \in H, x \in G.\end{aligned}$$

(ii) $h \cdot x = xh^{-1}$, $\forall h \in H$, $\forall x \in G$.
Naturally, $e \cdot x = xe^{-1} = x$, $\forall x \in G, e \in H$,
and
$$\begin{aligned}(ab) \cdot x &= x(ab)^{-1} = x(b^{-1}a^{-1}) = (xb^{-1})a^{-1} \\ &= (b \cdot x)a^{-1} = a \cdot (b \cdot x); \ \forall a, b \in H, x \in G\end{aligned}$$

(3) Let G be a group and H be a normal subgroup of G (i.e., $H \trianglelefteq G$). Define a map $\cdot : G \times H \to H$ by $g \cdot h = ghg^{-1}$, $\forall h \in H, \forall g \in G$.

Let $h \in H$. Now $e \cdot h = ehe^{-1} = ehe = h$, $e \in G$.

Let $g_1, g_2 \in G$, $h \in H$. Then

$$(g_1 g_2) \cdot h = (g_1 g_2) h (g_1 g_2)^{-1} = g_1 g_2 h g_2^{-1} g_1^{-1}$$
$$= g_1 (g_2 h g_2^{-1}) g_1^{-1} = g_1 (g_2 \cdot h) g_1^{-1}$$
$$= g_1 \cdot (g_2 \cdot h)$$

Hence $\cdot$ defines a group action of G on H, and H is a G-set.

Definitions: Let G be a group and S a set and $\cdot : G \times S \to S$ a group action.

(i) For each $x \in S$ the Orbit of x under G is defined as $O(x) = \{g \cdot x \mid g \in G\} \subset S$. OR $Orb_G(x)$

(ii) For each $x \in S$ the stabilizer of x in G is defined as $G_x = \{g \in G \mid g \cdot x = x\} \subseteq G$.

The stabilizer G_x is also known as the isotropy subgroup of x. $e \in G_x \Rightarrow G_x$ is non-empty.

(iii) The subset of S fixed by G is denoted by $S^G = \{x \in S \mid g \cdot x = x, \forall g \in G\}$. x is called here fixed by G.

Examples:

Consider the group action of G on itself ~~as~~ by conjugation as $g \cdot x = gxg^{-1}$, $\forall g, x \in G$.

i.e. $S = G$ and $g \cdot x = gxg^{-1}$, $\forall x, g \in G$.

Clearly, $e \cdot x = exe^{-1} = exe = x$, $\forall x \in G$.

Also for $a, b \in G$, $(ab) \cdot x = (ab) x (ab)^{-1} = a(bxb^{-1})a^{-1}$
$= a(b \cdot x)a^{-1} = a \cdot (b \cdot x)$, $\forall x \in G$

Now, the orbit of x is its conjugacy class:

$$O(x) = \{gxg^{-1} \mid g \in G\}.$$

The stabilizer of x in G is the <u>centralizer</u> of x

$$G_x = \{g \in G \mid g x g^{-1} = x\} = \{g \in G \mid gx = xg\}.$$

Because, the centralizer of x is the set of all group elements which commute with x.

The fixed subset of this group action is the centre of G:

$$S^G = \{g \in G \mid x g x^{-1} = g, \ \forall x \in G\}$$
$$= \{g \in G \mid xg = gx, \ \forall x \in G\} = Z(G).$$

<u>To show</u>: The stabilizer of x in G is a subgroup of G.

Let $x \in S$. Since $e.x = x$, where $e \in G$, then $e \in G_x \Rightarrow G_x$ is a non-empty subset of G.

Let $a, b \in G_x$. Then $a.x = x$, $b.x = x$.

$\therefore (ab).x = a.(b.x) = a.x = x$

$\Rightarrow ab \in G_x$

Also $a^{-1}.x = a^{-1}.(a.x) = (a^{-1}a).x = e.x = x$

$\Rightarrow a^{-1} \in G_x$.

$\therefore$ <u>$G_x \leq G$.</u>

<u>Theorem</u>: Let G be a group and S be a G-set. Define a relation $\sim$ on S by $\forall a, b \in S$

$a \sim b$ iff $g.a = b$ for some $g \in G$. (*) called the <u>Orbits</u>.

Then show that $\sim$ is an equivalence relation.

<u>Note</u>: The equivalence classes determined by this equiv. reln. are (*)

Since $\forall a \in S$, $e.a = a$, $a \sim a \Rightarrow$ Reflexive.

Let $a, b \in S$, and $a \sim b$. Then

$g.a = b$ for some $g \in G$.

$\Rightarrow g^{-1}.b = g^{-1}.(g.a) = (g^{-1}g).a = e.a = a$

$\Rightarrow b \sim a$ as $g^{-1} \in G$. $\Rightarrow$ Symmetric.

Let $a, b, c \in S$, and $a \sim b$, $b \sim c$. Then

$\exists \ g_1, g_2 \in G$ s.t. $g_1.a = b$, $g_2.b = c$.

Thus $c = g_2.b = g_2.(g_1.a) = (g_2 g_1).a$, $g_2 g_1 \in G$.

$\Rightarrow a \sim c$ for some $g_2 g_1 \in G$. $\Rightarrow$ Transitive.

$\therefore \sim$ is an equivalence relation.

Theorem: Let G be a group and S be a G-set. Then the group action of G on S induces a homomorphism from G onto $A(S)$, $A(S)$ being the group of all permutations of S.

Let $g \in G$. Define a map $T_g : S \to S$ by $T_g(a) = g.a, \forall a \in S$.

Let $a, b \in S$. Then

$$\left.\begin{array}{rl} & a = b \\ \text{[Since } \cdot : G\times S \to S \text{ is a map]} \Leftrightarrow & g.a = g.b \\ \Leftrightarrow & T_g(a) = T_g(b) \end{array}\right\} \therefore T_g \text{ is well-defined } (\Rightarrow) \text{ and one-one } (\Leftarrow).$$

Now $b = e.b = (gg^{-1}).b = g\cdot(g^{-1}.b) = T_g(g^{-1}.b)$, $g^{-1}.b \in S$.

$\Rightarrow$ for $b \in S$, $\exists\ g^{-1}.b \in S$ s.t. $T_g(g^{-1}.b) = b$

$\Rightarrow$ T_g is onto.

$\therefore$ T_g is a bijection and hence $T_g \in A(S)$.

Let $g_1, g_2 \in G$. Then

$$\begin{aligned} T_{g_1 g_2}(a) &= (g_1 g_2)\cdot a = g_1\cdot(g_2\cdot a) \\ &= T_{g_1}(g_2\cdot a) = T_{g_1}(T_{g_2}(a)) \\ &= (T_{g_1} \circ T_{g_2})(a), \ \forall a \in S. \end{aligned}$$

$\Rightarrow$ $T_{g_1 g_2} = T_{g_1} \circ T_{g_2}$

Let us define a map $\phi : G \to A(S)$ by

$\phi(g) = T_g, \ \forall g \in G$.

$\phi(g_1 g_2) = T_{g_1 g_2} = T_{g_1} \circ T_{g_2} = \phi(g_1) \circ \phi(g_2), \ \forall g_1, g_2 \in G$

$\Rightarrow$ ϕ is a homomorphism.

Note: In the above, we have shown that T_g is well-defined and one-one.

Where $a, b \in S$.

For well-defined: Let $a = b$,

Then $g.a = g.b$, [as $\cdot : G\times S \to S$ is a map & group action, $g.a \in S$ (codomain), $g.b \in S$ ("). i.e., images $g.a, g.b$ are equal for a map if $a = b$.]

$\Rightarrow$ $T_g(a) = T_g(b)$

$\therefore$ T_g is well-defined.

For one-one: Let $T_g(a) = T_g(b)$

$\Rightarrow g.a = g.b \Rightarrow g^{-1}(g.a) = g^{-1}(g.b)$

$\Rightarrow (g^{-1}g).a = (g^{-1}g).b$

$\Rightarrow$ $e.a = e.b \Rightarrow a = b \Rightarrow T_g$ is one-one.